Ratios, Percentages, and Rates

We deal with mathematical information every day, mostly in the form of ratios, percentages, and rates. What is the chance of rain today? What is the interest rate for a loan? What is the price of gasoline? What grade did you get on the last math test? How large a tip should you leave at a restaurant? We use ratios in cooking and chemistry, and to describe batting averages. Polling data, nutritional facts, and sale prices are expressed in percentages. We pay fixed rates for water, phone calls, electricity and taxes.

Ratios, percents, and rates are useful because they are intuitive. Of course, that intuition has to be *learned*. In this chapter we study how the Primary Mathematics curriculum cultivates this intuition through the use of bar diagrams and simple logic. We begin all three topics by asking *"What is the unit of measurement?"*

7.1 Ratios and Proportions

Measurements make sense only when a unit is specified. One can say "the book weighs 5 pounds", but the statement "the book weighs 5" is meaningless until we are told which unit of weight has been used. Measurements yield *quantities* — numbers with units — such as 5 pounds, $2\frac{1}{3}$ cups, or 8.21 seconds.

quantity

Nevertheless, one can *compare* measurements without specifying a unit. For example, suppose Carla is twice as tall as her little brother. It does not matter whether their heights are measured in inches, centimeters, or pencil lengths. Twice as tall is twice as tall. Any unit can be used provided that *the same unit is used for both measurements*. That simple observation explains what ratios are and why they are useful.

> **SCHOOL DEFINITION 1.1.** *We say that the **ratio** between two quantities is $A : B$ if there is a unit so that the first quantity measures A units and the second measures B units. (In writing the ratio one does not specify the unit.)*

Determining a ratio is very easy: we simply make measurements using some unit and record the result, taking care to specify the order of the measurements, but deliberately leaving out all information about the unit of measurement. A beautiful introduction to this process is given on pages $71 - 74$ of Primary Math 5A. Have a look!

EXAMPLE 1.2. *If orange juice is made by mixing 5 cans of water with 1 can of concentrate, then the water-to-concentrate ratio is* $5 : 1$.

Notice that the ratio tells us nothing about the size of the can (which was the unit of measurement). Also notice the importance of specifying the order: while the water-to-concentrate ratio is $5 : 1$, the concentrate-to-water ratio is $1 : 5$. We would not want to confuse the two!

It is sometimes useful to consider ratios of three quantities.

EXAMPLE 1.3. *The crowd at a basketball game consisted of 400 men, 200 women, and 300 children. Thinking in units of 100 people, the ratio of men to women to children was* $4 : 2 : 3$.

In any measurement we are free to choose the units. In the above example we took "100 people" as the unit. But we could equally well have taken the unit to be one person and said the ratio is $400 : 200 : 300$. This second ratio describes exactly the same physical situation; each of the numbers is 100 times larger simply because we have switched the unit of measurement. That observation leads to the notion of equivalent ratios.

DEFINITION 1.4. *Two ratios are* **equivalent** *(are "equal ratios") if one is obtained from the other by multiplying or dividing all of the measurements by the same (nonzero) number.*

Thus the ratio $6 : 10$ is equivalent to $18 : 30$ (multiplying by 3) and also equivalent to $3 : 5$ (dividing by 2). Of these, the ratio $3 : 5$ is in "simplest form" because the numbers 3 and 5 have no common factor.

ratio in simplest form

EXERCISE 1.5. *Read pages 71-76 of Primary Math 5A. These pages introduce ratios. Notice the rapid transition from concrete to pictorial to abstract.*

a) *On page 75, how are boxes used to illustrate how changing the size of the units creates an equivalent ratio?*

b) *On page 76, the idea of the simplest form is explained. The student helpers in the margins show how to find the simplest form of a ratio by dividing both numbers by* _____ .

The process of writing down a ratio cannot be reversed: knowing the ratio is not, by itself, enough to determine the actual measurements. The ratio gives only partial information.

The key to using ratios is seeing how to combine the ratio with other information *to determine a unit*. Once you know the unit, everything else follows.

EXAMPLE 1.6. *The ratio of boys to girls in the class is 5 : 3. There are 6 more boys than girls. How many students are there in the class?*

The ratio 5 : 3 was obtained by measuring using the same unit. To solve the problem we draw a diagram and determine that unit.

Boys

Girls

?

6

2 units = 6
1 unit = 3
8 units = 24
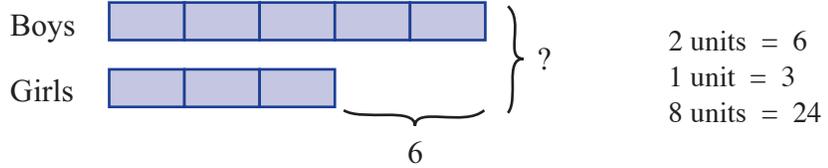
There are 24 children.

EXAMPLE 1.7. *The ratio of Jim's money to Peter's money was 4:7 at first. After Jim spent $\frac{1}{2}$ of his money and Peter spent $60, Peter had twice as much money as Jim. How much money did Jim have at first?*

?

Jim

Peter

$60

3 units = $60
1 unit = $20
4 units = $80
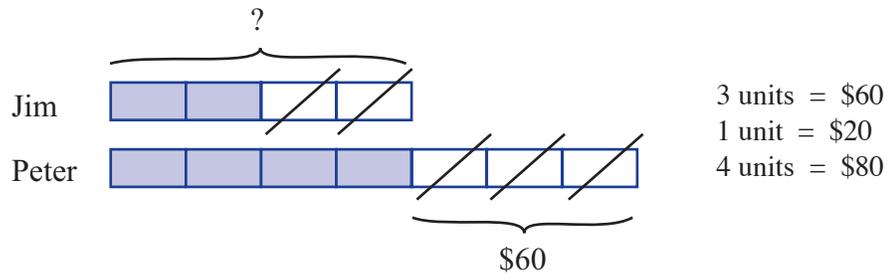
Jim started with $80.

Proportions

A blueprint of a house allows us to visualize the shape and relative size of the rooms. The blueprint is a scale drawing — the measurements in the blueprint are in a fixed ratio to the actual measurements. Knowing that ratio, we can use the blueprint to determine the size of each room.

scale

For example, if an inch on the blueprint corresponds to 60 actual inches, we say the *scale* is the ratio 1 : 60. If a wall is 2 inches long on the blueprint, the length of the actual wall (in inches) is found by solving the problem

$$1 : 60 = 2 : \underline{\quad}.$$

Such an equation is called a proportion.

DEFINITION 1.8. *A **proportion** is a statement that two ratios are equal.*

EXAMPLE 1.9. *The ratio of girls to boys in a class is 4:5. If there are 12 girls in the class, how many boys are there?*

This question can be written as

$$4 : 5 = 12 : \underline{\hphantom{--}}.$$

Recall what it means for these ratios to be equal: there is some number which when multiplied by 4 gives 12 and when multiplied by 5 gives ___. To get from 4 to 12 we multiply by 3, so the unknown number is $5 \times 3 = 15$. There are 15 boys in the class.

EXERCISE 1.10. *The ratio of Isabel's money to Rosalind's money is $8 : 3$. If Isabel has $24, how much money do the two girls have together?*
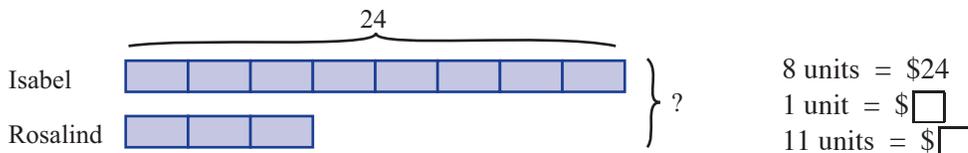
Here are three different Teacher's Solutions to Exercise 1.10.

- *Using proportions.* If Rosalind has R dollars, then the statement of the problem implies $8 : 3 = 24 : R$. As you will show in Homework Problem 6, the proportion $8 : 3 = 24 : R$ is equivalent to
$$\frac{8}{3} = \frac{24}{R},$$
or $8R = 3 \times 24$. Solving shows that Rosalind has $R = 9$ dollars, so Isabel and Rosalind together have $\$24 + \$9 = \$33$.

- *Using a bar diagram.*

They have $\$\square$ altogether.

- *Using algebra.* The bar diagram also shows how to solve the problem using algebra. Let x stand for the amount of money in 1 unit in the bar diagram above. Then Isabel has $8x$ dollars and Rosalind has $3x$ dollars. The solution using algebra then goes as follows.

  Solution: By the definition of ratio, there is a number x so that Isabel's money is $\$8x$ and Rosalind's money is $\$3x$. Hence

  $$\begin{aligned} 8x &= 24 \\ x &= 3 \\ \text{Total: } 11x &= 11 \cdot 3 = 33 \end{aligned}$$

  They have $33 altogether.

Thus far, bar diagrams have been our main tool for solving word problems. Yet two of the three solutions above use prealgebra methods. In fact, after using bar diagrams to familiarize

students with the idea of ratios, the Primary Mathematics curriculum moves on to an algebraic approach to ratios in 7th grade. There are two reasons for this shift. First, the topic of ratios occurs at that point in the curriculum (the end of elementary school) where the mathematical content switches from an emphasis on arithmetic to an emphasis on algebra. Second, bar diagrams are not always the best approach for solving ratio problems, as the following example shows.

EXAMPLE 1.11. *Two numbers are in the ratio 3:5. After subtracting 11 from each, the new ratio is 2:7. What are the two numbers?*

This problem appears to be simple. But you will have trouble creating a bar diagram for it (try!). An algebraic solution is easier.

Teacher's Solution using algebra:

Since the ratio is $3 : 5$, there is a number x such that first number is $3x$ and the second is $5x$. After subtracting 11, the ratio becomes $(3x - 11) : (5x - 11)$. Since that is equal to $2 : 7$, we get the proportion

$$\frac{3x - 11}{5x - 11} = \frac{2}{7}.$$

Now solve:

$$21x - 77 = 10x - 22,$$

$$11x = 55,$$

$$x = 5.$$

The first number is $3x = 15$ and the second is $5x = 25$.

Ratios and fractions

Ratios are not numbers — one cannot add, subtract, multiply or divide ratios. For instance, how would you multiply the following? Is there even a plausible interpretation for doing so?

$$(3 : 2 : 7) \times (8 : 4 : 5) = ?$$

Nevertheless, ratios can often be usefully converted into fractions *if one is careful to specify the whole unit*. In any given situation that can be done in several ways.

EXERCISE 1.12. *Read Problem 6 on page 24 of Primary Math 6A. In that problem the ratio $2 : 5$ is converted into four different fractions depending upon the choice of a whole unit. Observe:*

- *In part c) the ratio $2 : 5$ is converted into the fraction statement "The length of A is $\frac{2}{7}$ of the total length". The whole unit is the total length.*

- *In part e) the ratio $2 : 5$ is converted into the fraction statement "The length of A is $\frac{2}{5}$ of the length of B". The whole unit is the length of B.*

- *In part f), what choice of whole unit leads to the fraction $\frac{5}{2}$?*

It is also sometimes useful to regard fractions as ratios.

EXERCISE 1.13. *Read and answer Problem 12 on page 26 of Primary Math 6A.*

Regarding ratios as numbers can lead to trouble, as the following example shows.

EXAMPLE 1.14. *A bag contains 6 white and 10 red marbles. Then 4 white marbles and 20 red marbles are added to the bag. What is the new ratio of white to red marbles?*

Diana, Kevin, and Mary came up with the following different solutions.

- Diana reasoned that the initial ratio was $6 : 10$, or $3 : 5$, and the added marbles had a ratio of $4 : 20$ or $1 : 5$. She then "added ratios", writing

$$3 : 5 \, + \, 1 : 5 \, = \, 4 : 10.$$

After simplifying, she concluded that the new ratio was $2 : 5$.

- Kevin converted the information "6 white and 10 red marbles" to the fraction $\frac{6}{10} = \frac{3}{5}$ and the information "4 white and 20 red marbles" to the fraction $\frac{4}{20} = \frac{1}{5}$. He then wrote

$$\frac{3}{5} \, + \, \frac{1}{5} \, = \, \frac{4}{5}$$

and concluded that the new ratio was $4 : 5$.

- Mary simply counted marbles: in the end there were $6 + 4 = 10$ white and $10 + 20 = 30$ red marbles, so the new ratio was $10 : 30$ or $1 : 3$.

Who is right? And what did the other two do incorrectly? (See Homework Problem 7.)

Homework Set 30

1. *(Study the textbook!)* Read pages $71 - 78$ of Primary Math 5A, doing the problems mentally as you read. In Practice 5A, answer Problems $1 - 4$ and give Teacher's Solutions for Problems $5 - 7$.

2. *(Study the textbook!)* Read pages $79 - 81$ of Primary Math 5A, doing the problems mentally as you read. In Practice 5B, answer Problems $4 - 7$ and give Teacher's Solutions for Problems 8 and 9.

3. Draw a picture illustrating why the ratios 12:16, 6:8 and 3:4 are equal (see page 75 of Primary Math 5A).

4. In **Workbook 5A**, read and answer problems on pages $82 - 90$ by filling in the answers in the workbook (do not copy onto your homework paper).

5. *(Study the textbook!)* Read pages $21 - 33$ of Primary Math 6A, doing the problems mentally as you read. Give Teacher's Solutions for Problems $6 - 8$ of Practice 3A and Problems $6 - 9$ of Practice 3B.

6. If $a : b = c : d$, prove that a) $\dfrac{a}{b} = \dfrac{c}{d}$ and b) $ad = bc$.

   Hint: By Definition 1.4, if the ratios $a \, : \, b$ and $c \, : \, d$ are equivalent then there is a (nonzero) number x so that $a = cx$ and $b = dx$.

7. Reread the three student solutions in Example 1.14 above. Who is right, and what did the other two do incorrectly? (*Hint:* it will help to be very clear about units. Diana's ratio $3 : 5$ is a count using what unit? Her ratio $1 : 5$ is a count using what unit? Kevin's $\frac{3}{5}$ means $\frac{3}{5}$ of what unit?)

8. A class is presented with the following problem.

In a bag of marbles, the ratio of white marbles to red marbles is $2 : 3$ *and the ratio of red to black marbles is* $6 : 11$. *What is the ratio of white to black marbles?*

Conner, who likes using fractions, writes $\frac{2}{3} \times \frac{6}{11} = \frac{4}{11}$ and announces, without explanation, that the ratio of white to black marbles is $4 : 11$. Explain why he is right. (*Hint: taking the number of black marbles to be the whole unit, what fraction are red, and then what fraction are white?*)

7.2 Changing Ratios and Percentages

We begin this section by considering problems involving changing ratios. We then start the next topic: percentage. Bar diagrams are valuable aids for both types of problems.

In a "changing ratio" problem one is presented with a before-and-after situation. The problem is to relate the measurements and ratios before the change to measurements and ratios after the change. Such problems can be illustrated by drawing separate before and after bar diagrams.

EXAMPLE 2.1. *The ratio of the number of Jason's marbles to Tom's is* $3 : 5$. *Jason has 42 marbles. If Jason buys another 8 marbles, what will be the new ratio of Jason's marbles to Tom's?*

Before:

3 units $=$ 42
1 unit $=$ 14
5 units $=$ 70.

After:

$50 : 70 = 5 : 7$
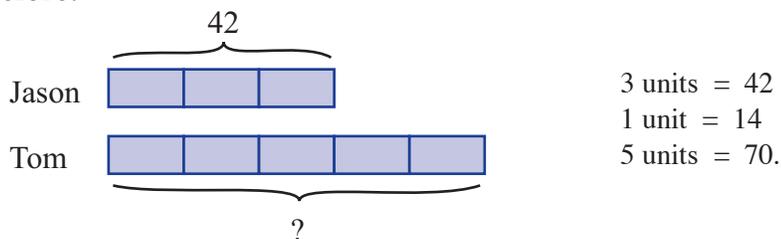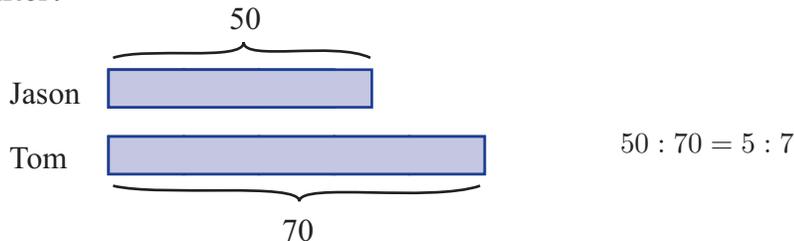
The new ratio of Jason's marbles to Tom's is 5:7.

EXERCISE 2.2. *Before proceeding, read pages 34–37 of Primary Math 6A, noting how the bar diagram is constructed in each example.*

In the next problem it becomes important to record the amount removed. This helps to determine the unit, which is the key to solving the problem.