

Geometric Figures

This chapter describes how elementary students are introduced to the world of geometry. We have seen how children learn to measure lengths and angles and to solve arithmetic problems with measurements. At the same time, in a separate part of the curriculum, geometry begins as a subject in its own right.

Geometry is the study of relationships among the measurements – lengths, angles, areas and volumes – of figures. Already by grade 2, geometry moves beyond naming figures: the activities direct attention to lengths and angles. In grades 2–4, children learn about parallel and perpendicular lines and solve problems involving supplementary and vertical angles. They also learn to draw figures using a ruler, protractor, compass and set square. In grades 5 and 6 the pieces come together and they begin solving problems involving lengths and angles within triangles and quadrilaterals.

The elementary geometry curriculum focuses on clear reasoning and simple geometric facts. There are no proofs; instead, facts are introduced using paper-folding or symmetry arguments. Reasoning skills are developed through daily problem sets, problem sets that can be great fun for both children and teachers.

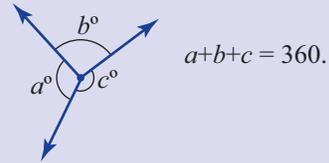
2.1 Fundamental Geometric Ideas

As children learn to measure in grades K–3 they acquire intuition about segments, angles and other objects of geometry. Sometime near the end of elementary school, intuition is replaced by precise definitions. This section describes how angles, perpendicular and parallel lines, and figures are presented intuitively, and then in the precise but child-friendly form of a “school definition”.

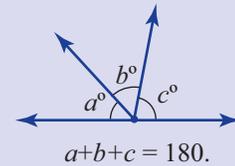
Angles

The “angle measurement facts” listed on page 24 have three especially useful consequences:

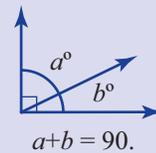
1. The total measure of adjacent angles around a point is 360° . (Abbreviation: \angle s at a pt.)



2. The total measure of adjacent angles forming a straight line is 180° . (Abbreviation: \angle s on a line.)



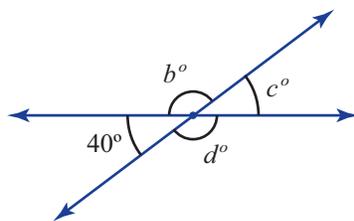
3. The sum of adjacent angles in a right angle is 90° . (Abbreviation: \angle s in rt. \angle .)



As you will see, these facts will be used repeatedly; they are the springboard that launches deductive geometry in grades 5 and 6. We have given each an abbreviation. Learning geometry is easier in classrooms where everyone consistently uses the same short, clear abbreviations.

When two lines intersect, they form four angles around the point of intersection. If we know the measure of any one of these, we can determine the measures of all four by recognizing pairs of supplementary angles.

EXAMPLE 1.1. In the figure, find the measures of angles b , c and d .



Solution: Moving around the vertex clockwise, we see successive pairs of supplementary angles:

$$40 + b = 180, \text{ so } b = 140.$$

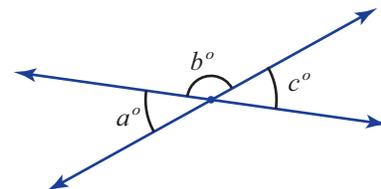
$$b + c = 140 + c = 180, \text{ so } c = 40.$$

$$c + d = 40 + d = 180, \text{ so } d = 140.$$

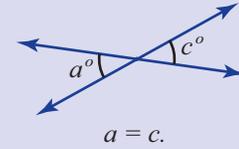
vertical angles

In the figure below, angles a and c are a pair of *vertically opposite angles* because they are opposite each other through the vertex. For short, one says that a and c are *vertical angles*. Teachers should avoid the phrase “opposite angles” because pairs of angles that might be called “opposite” occur in other, different contexts (see page 48).

The figure contains two pairs of supplementary angles. Angles a and b are angles on a line, so a is $180 - b$. Similarly, c is also $180 - b$. Therefore $a = c$, so the two vertical angles have equal measure.



Vertical angles have equal measure.
(Abbreviation: **vert.** \angle s.)



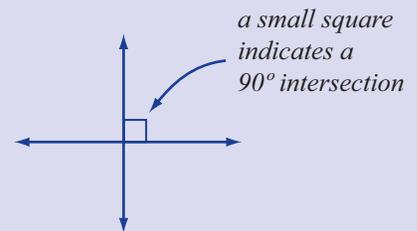
EXERCISE 1.2. Read pages 85–88 in *Primary Math 5A*. Which angle facts are introduced on these pages?

Perpendicular and Parallel Lines

As we saw in Section 1.4, students are introduced to right angles in *Primary Math 3B*. Then in *Primary Math 4A*, they learn that a right angle measures 90° and, a few pages later, they learn the term “perpendicular”.

DEFINITION 1.3. Two segments, rays, or lines are **perpendicular** if the lines containing them intersect to form a 90° angle.

If \overleftrightarrow{AB} is perpendicular to \overleftrightarrow{CD} , we write $\overleftrightarrow{AB} \perp \overleftrightarrow{CD}$.

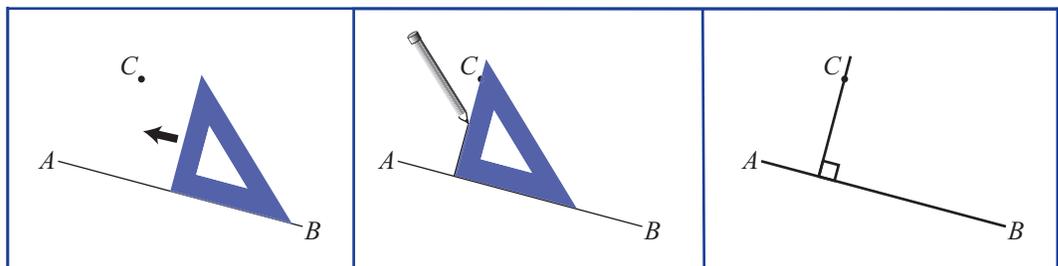


Two intersecting lines form four angles. If one of those angles is 90° , then by symmetry each of the other angles must be 90° .

Drawing perpendicular lines is harder than recognizing them because it requires motor skills. Hands-on activities drawing perpendicular lines deepen students’ understanding and develops skills that will be useful later. In classrooms, right angles are usually drawn with the aid of a *set square* (or *plastic triangle*), although any object with a right angle such as a piece of cardboard can be used.

set square

EXAMPLE 1.4. Use a set square to draw a perpendicular to the line \overleftrightarrow{AB} through the point C .



parallel lines

Two lines are *parallel* if they lie in the same plane and do not intersect. In addition, in K-8 geometry — but not always in high school geometry — a single line is considered parallel to itself. When lines \overleftrightarrow{AB} and \overleftrightarrow{CD} are parallel, we write $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$. In pictures, pairs of matching arrows are used to indicate that two lines are parallel.



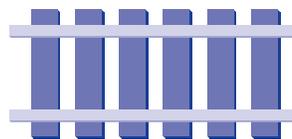
parallel segments

Two segments are parallel if they are part of parallel lines.

As always, the word “line” means a straight line that extends indefinitely in both directions. Consequently, to make sense of the phrase “do not intersect” children must envision extending the segments on their paper indefinitely into space, past distant galaxies — something that is not very concrete. Furthermore, the condition that two lines do not intersect does not suggest a way of drawing parallel lines, or a way to determine when two segments are parallel. Consequently, a different definition of parallel lines is used in elementary school.

DEFINITION 1.5 (School Definition). *Two lines, segments, or rays are **parallel** if they lie in the same plane and are both perpendicular to a third line.*

This school definition gives a mental picture that is concrete and easily explained. Examples of segments that have a common perpendicular are all around us, while there are few examples of non-intersecting lines extending indefinitely into space past distant galaxies.

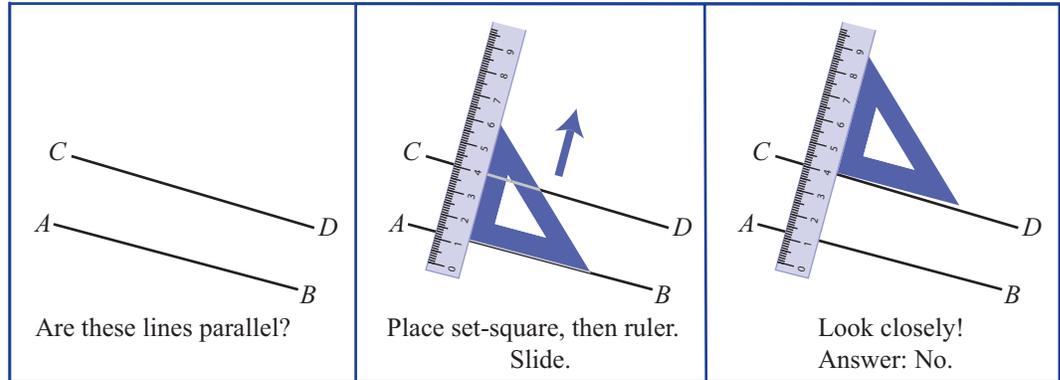


Railroad tracks and lined paper have many parallel lines.

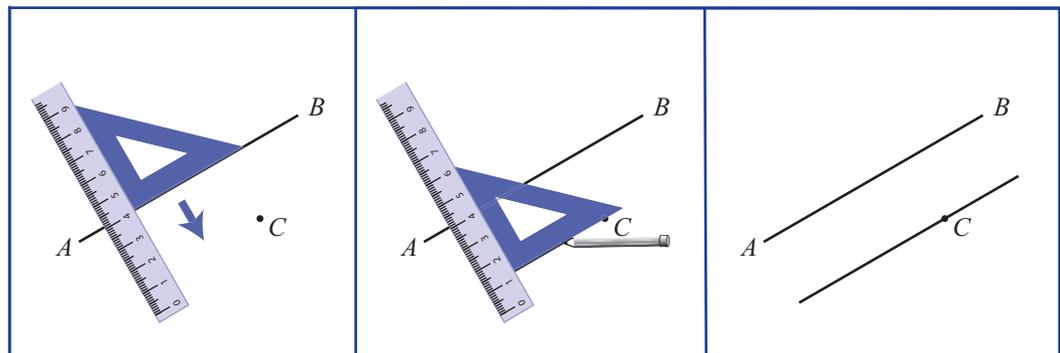


After describing parallel lines, the Primary Mathematics curriculum focuses on two activities to help students develop intuition about parallel lines: a method for determining whether two lines are parallel and a method for constructing parallel lines using a set square. The first method calls children’s attention to a fact that is treated as common knowledge in elementary mathematics: if two lines are both parallel to a third line, then they are parallel to each other.

EXAMPLE 1.6. Use a set square to determine whether two lines are parallel.



EXAMPLE 1.7. Use a set square and a ruler to draw a line parallel to \overleftrightarrow{AB} through the point C .

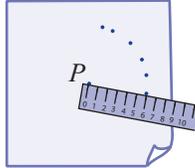
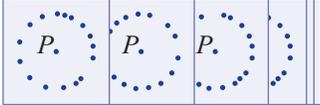
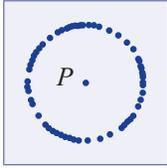


The Primary Math books include a marvelous set of problems involving parallel lines within parallelograms and other figures (see Section 2.4). Then in seventh grade, students study perpendicular and parallel lines more abstractly.

Circles

Children learn to recognize and name circles in kindergarten and first grade, but it is usually not until third or fourth grade that they encounter the precise definition of a circle. At this time they learn the terms “radius” and “diameter”, and learn to draw circles with compasses. Here is a three-step teaching sequence for introducing circles:

Step 1 — Definition. Circles are defined in terms of distance, not visual shape. The following activity helps children understand the key idea.

 <p>Mark point P on a transparency. Then mark as many points as you can that are 6 cm from point P.</p>	 <p>Put your transparency on top of your classmates' papers.</p>	 <p>What do you notice?</p>
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This activity shows that (i) a circle is a collection of points, and (ii) a circle is completely determined by its center and its radius. With these two realizations, children understand the essence of the mathematical definition

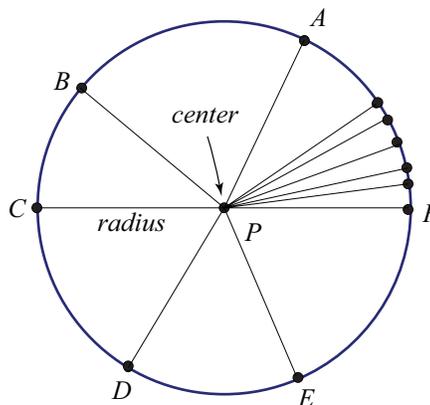
circle
center
radius

DEFINITION 1.8. Choose a point P in the plane and a distance R . The **circle** with center P and radius R is the set of all points in the plane that are distance R from the point P .

radius

The word “radius” has two meanings. *The radius* of a circle is a distance, as in Definition 1.8, while *a radius* is any segment with one endpoint on the center and the other endpoint on the circle. (The plural of this word is “radii”, pronounced “ray-de-eye”.) Because double meanings engender confusion, alert teachers always clarify whether “radius” refers to a distance or a segment.

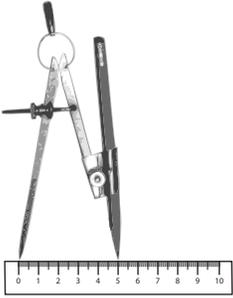
In a circle *all radii have the same length*. Teachers can reinforce this aspect of the definition by having students use rulers to check that all radii have the same length.



compass

Step 2 — Drawing Circles. A *compass* is a tool for drawing circles. Learning to use a compass is a prerequisite for later geometry.

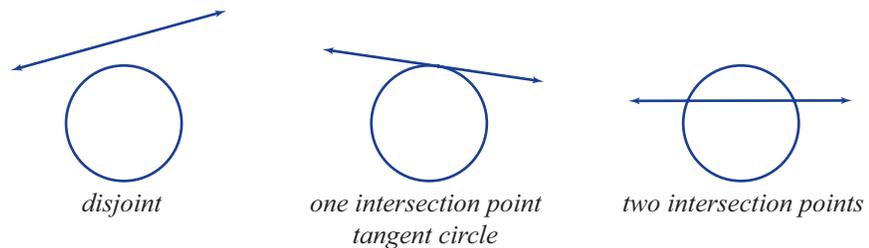
EXERCISE 1.9. Mark a point P on your paper. Draw a circle with center P and radius 5 cm.

 <p>Adjust your compass to 5 cm.</p>	 <p>Place the compass point on P. Rotate to draw a circle.</p>	<p>Use one hand, not two.</p> <p>Lean the compass forward and pull - don't push - the pencil.</p> <p>Don't let the point of the compass slip off P.</p>
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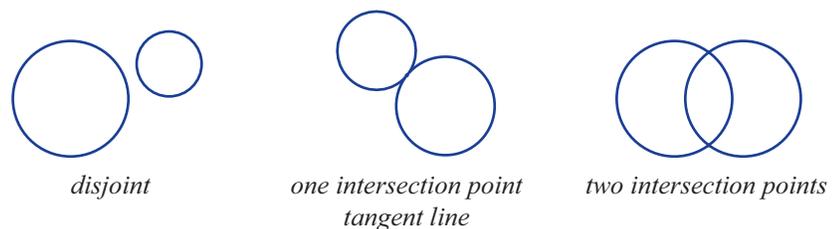
As in Definition 1.8, a circle is determined by choosing a center point and choosing a radius. These are exactly the choices students make when they use a compass to draw circles. Other methods for drawing circles, such as tracing around a tin can, hide those choices. The simplicity of the definition gets lost!

Using a compass to draw circles is easy, but requires some practice. The quality of the compass is a factor here. Compasses with screw adjustments work well, but the arms of low-quality compasses tend to slip, making it frustratingly difficult to maintain a constant radius while drawing circles.

Step 3 — Properties of Circles. A line and a circle can intersect in zero, one, or two points.



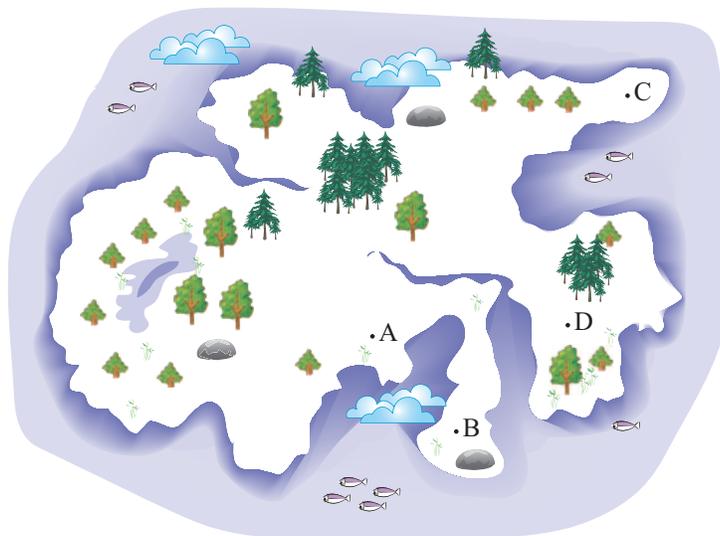
Likewise, two different circles intersect in zero, one, or two points.



EXERCISE 1.10. Use a compass and a straightedge to draw two circles that are tangent. (Hint: Draw a circle with center P and a ray with endpoint P . How can you choose a center point and a radius for a second, tangent circle?)

EXERCISE 1.11. *The following third grade problem describes circles that intersect in two points, but the problem has only one solution. Why?*

Gold is buried 4 cm from point B and 2 cm from point C. Use your compass to find the treasure.



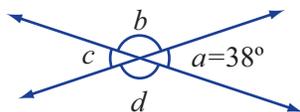
The teaching sequence continues in Primary Mathematics textbooks in sixth grade with definitions and applications of terms like diameter, circumference, and semicircle (see pages 26–29 of Primary Math 6B). We will examine this phase of the curriculum in Chapter 8.

Homework Set 5

- For each of the following times of day, sketch a clock face showing that time and find the measure (in degrees) of the angle formed by the hour hand and the minute hand. (Remember that the hour hand moves 30° per hour.)

a) 3:00 b) 3:30 c) 10:30 d) 2:45.

- In the figure, $a = 38^\circ$. Find b , c , and d .

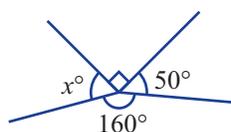


- On page 238 of NEM1, do Problem 1 of Class Activity 2.
- (Study the textbook!) Read the rest of Class Activity 2. These exercises have students accept three basic facts after checking them in one example. These are not proofs; the purpose of the activity is to make clear *what the facts*

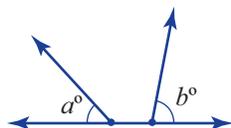
mean and help make them evident to the students (rather than being teacher-announced truths).

- Copy down the three facts with their abbreviations. We will shorten the first two abbreviations to “ \angle s on a line” and “vert. \angle s”.
 - Are these the same three facts introduced on pages 85–86 of Primary Math 5A?
- In Exercise 9.2 on pages 240–241 of NEM1, do problems 1b and parts b), f), h) and j) of Problem 2. Write your solutions in the manner of Worked Examples 1 and 2 on pages 239 and 240 on NEM 1. Be sure to include reasons in parentheses.
 - (Study the textbook!)
 - Read pages 78–83 of Primary Math 4A. Does the book define the terms perpendicular and parallel, or does it just show examples?
 - Now read pages 242–243 of NEM1. Does NEM1 define the terms perpendicular and parallel?

- c) Give a one or two sentence explanation of how Problem 2 on page 80 of Primary Math 4A helps students make sense of the seventh grade definition of perpendicular.
- d) Give a similar explanation of how Problem 2 on page 83 helps students make sense of the seventh grade definition of parallel lines.
7. On page 243 of NEM1, do Problem 1 of Class Activity 3 using a ruler and a set square. Then write a precise definition for the term *perpendicular bisector*.
8. (*Common Student Error*) When asked to find x in the figure below, Mary writes “ $x = 50$ ” and explains “because vertical angles are equal”. What is x actually? What erroneous assumption did Mary make?



9. (*Common Student Error*) In the figure, Jerry claims that “ $a + b = 180$ ” stating as the reason, “ \angle s on a line.”



There may be several reasons why Jerry is making this error. He may not understand angle measurements or how

to add angle measurements, he may not understand that to apply this fact the angles must be _____, or he may not understand that there are _____ degrees on a straight angle.

10. Draw a line L and a point P not on L . Using a ruler and set square, follow the procedure of Example 1.7 in this section to draw a line through P parallel to L .
11. A teacher asks her students for a precise definition of the term *circle*.
- Sarah says “A circle is a round segment with no endpoints”. Name at least two things wrong with Sarah’s definition.
 - Michael says “A circle is 360° ”. What two notions is Michael confusing?
 - Write down a precise definition of the term *circle*.
12. Practice drawing circles with a compass until you can draw complete circles without stopping. Then on your homework paper draw circles of radius 2 cm, 6 cm and 10 cm, using your ruler to set the compass width.
13. Use your compass to draw a circle; label your circle C and its center O . Choose any point on the circle and label it P . Draw another circle of radius OP with center P .
14. Use your compass to draw circle B ; label its center O . Choose any point on the circle and label it Q . Draw two more circles of different radii such that both circles intersect circle B only at point Q . (Hint: Draw line \overleftrightarrow{OQ} first.)

2.2 Triangles

This section explains how an elementary school teacher might introduce one of the most important ideas in geometry: the fact that the sum of measures of the interior angles in any triangle equals 180° . To set the stage, we briefly discuss how children are introduced to shapes and figures in grades K-2.

Geometric Figures in the Early Grades

Pre-school, kindergarten, and first grade children learn to name, identify and draw 10 – 15 types of figures in the plane. There are two distinct levels to this learning. At the lower level, students identify shapes by sight-recognition: a square is a shape that *looks* like a square. Shape-naming is easy because young children have an innate ability to recognize and match shapes and are dazzlingly quick at learning new vocabulary words. At the higher level, students learn a precise definition and learn to check that the figures satisfy the required conditions. A “hexagon”, for example, is not simply something that looks like a hexagon. Rather, a hexagon is a figure with six straight sides — count ‘em to be sure! The teacher’s role is to move students from the first level to the second.